**Dynamic Model for a Planar Robotic Arm with Two Rotational Degrees of Freedom**

Here we compute the closed form dynamic equations for a two-link planar manipulator. Let's denote the angles of the two joints as and .

It is noted that the following is based on [1].



We assume that all mass exists as a point mass at the distal end of each link. The masses of link 1 and link 2 are and , respectively. The mass of the load at the end of the arm is and as it is also at the end of link 2 we assume:

The lengths of link 1 and link 2 are and , respectively.

The viscous friction coefficients at the two joints are c1 and c2, respectively.

The gravitational acceleration is:

(2)

Because of the point-mass assumption, the inertia tensor written at the center of mass for each link is the zero matrix.

The base of the robot is not rotating, hence we have:

The vector of joint angles is:

We assume that there are no external forces or torques acting on the system. We also assume that the robot links and joints are rigid.

**The state space equation**

When the Newton-Euler equations are evaluated symbolically for a two-link manipulator, they yield a dynamic equation that can be written in the form:

where is the 2X2 inertia matrix of the manipulator:

where are moments of inertia of the joints about their respective rotational axes.

are cross products of inertia of the joints.

Any manipulator inertia matrix is symmetric and positive definite, and is, therefore, always invertible.

**V(, )** is a 2X1 vector of centrifugal and Coriolis terms:

where v1 represents the Coriolis and centrifugal effects due to joint 1's motion and v2 represents the Coriolis and centrifugal effects due to joint 2's motion.

We use the term state-space equation because the term **V(, )**, appearing in (6), has both position and velocity dependence.

We assume that the motion of one joint is independent of the motion of the other joint.

is a 2X1 vector of gravity terms:

We assume that the gravitational forces act independently on each joint.

Note that the gravity term depends only on θ and not it's derivatives.

**F(Θ)** is the 2X1 viscous friction vector:

where b1 represents the viscous friction coefficient of joint 1 and b2 represents the viscous friction coefficient of joint 2.

The friction effects are commonly difficult to model. Here, we assume a viscous friction that is according to eq. 20 and 21. In the assumed model for the control we will use the linear version, whereas in the simulation of the actual we will not.

for the assumed model.

**τ** is the 2X1 vector of torques applied at the joints:

**Computed Torque based Control (Feedback linearization)**

**The Theoretical Case**

This nonlinear control technique is a model-based control. When eq. 6 is an exact model of our MIMO system, this results in a set of decoupled linear control equations – one per each joint. This is derived as follows:



As shown in the scheme, in this case we choose:

(23)**τ** = **ατ' + β**

where τ is the 2X1 vector of joint torques. We choose

**α** = **M()**,

**β** = **V(, ,**

Also, as shown in the scheme, the servo law is:

**τ'** =

Using (5) and (23) through (26), it is possible to show that the closed-loop system is characterized by the error equation

Note that this vector equation is decoupled in the case that the matrices Kv and Kp are chosen to be diagonal, so that (28) could just as well be written on a joint-by-joint basis as

i + kvi + kpie = 0 (29)

The ideal performance represented by (28) is unattainable in practice, for many reasons, the most important two being:

1. The discrete nature of a digital computer implementation, as opposed to the ideal continuous time control law implied by (24) and (25).
2. Inaccuracy in the manipulator model (needed to compute (24)).

**The difference between the arm simulation and the controller model**

In the arm simulation we consider the load mass which means this simulation represents accurately the physical problem presented here. However, in the controller model we neglect the mass of the load to get a simplified representation of the system's dynamics while formulating control laws.

**References**

[1] John J. Craig. Introduction to Robotics Mechanics and Control. Third Edition. New Jersey, Pearson Education International , 2005.

**Simulating a Non-Theoretical Case**

In our case we will use a model for the actual arm with a non-linear viscous friction model and an assumed linear model for the model-based control. This means that the derivation of the theoretical case is no longer valid.